RAMAKRISHNA MISSION VIDYAMANDIRA

(Residential Autonomous College under University of Calcutta)

B.A./B.SC. SIXTH SEMESTER EXAMINATION, MAY 2015

THIRD YEAR

Date : 27/05/2015 **MATHEMATICS (Honours)**

Time : 11 am – 3 pm **Paper :** VII Full Marks : 100

[Use a separate Answer Book for each Group]

Group – A

Unit – I

[*Answer **any three** questions*]

1. a) Find the Fourier series of the periodic function f with period 2π defined as $f(x)=e^{x}-e^{-x}$, $x\in[-\pi,\pi]$ and hence show that, $\frac{1}{1^2+1}-\frac{3}{3^2+1}+\frac{5}{5^2+1}-\frac{7}{7^2+1}+...=\frac{\pi}{4\cosh\frac{\pi}{2}}$. [4]

   b) State the Dirichlet's conditions for expansion into Fourier Series of a function. [3]

   c) Show that for all values of x in $[-\pi,\pi]$, $\cos kx=\frac{\sin k\pi}{\pi}\left[\frac{1}{k}+\sum_{n=1}^{\infty}\frac{(-1)^n 2k\cos nx}{k^2-n^2}\right]$ where $0<k<1$. [3]

2. a) By changing the order of integration, prove that $\int_0^1 dx\int_0^{\sqrt{1-x^2}}\frac{dy}{(1+e^y)\sqrt{1-x^2-y^2}}=\frac{\pi}{2}\log\left[\frac{2e}{1+e}\right]$. [4]

   b) Evaluate $\iint_E y\,dx\,dy$ over the region E in the first quadrant bounded by x-axis, the curves $x^2+y^2=a^2$, $y^2=bx$. [2]

   c) Show that $\iint_E x^{1/2}y^{1/3}(1-x-y)^{2/3}dxdy=\frac{\Gamma\left(\frac{5}{3}\right)\Gamma\left(\frac{4}{3}\right)\Gamma\left(\frac{3}{2}\right)}{\Gamma\left(\frac{9}{2}\right)}$ where E is the triangle bounded by $x=0$, $y=0$ and $x+y=1$. [4]

3. a) Find the surface area of the part of the plane $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$ which lies between the co-ordinate planes. [2]

   b) Show that the volume of the solid bounded by the cylinder $x^2+y^2=2ax$ and the paraboloid $x^2+z^2=4ax$ is $(3\pi+8)\frac{2a^3}{3}\cdot(a>0)$. [4]

   c) Prove that $\iiint_E\frac{dx\,dy\,dz}{\sqrt{1-x^2-y^2-z^2}}=\pi^2$ where E is the region bounded by the interior of the sphere $x^2+y^2+z^2=1$. [4]

4. a) Show that $\int_0^{\pi/2}\frac{x^m}{\sin^n x}dx$ is convergent if and only if $n<1+m$. [3]

   b) Prove that $\sqrt{\pi}\,\Gamma(2x)=2^{2x-1}\Gamma(x)\Gamma\left(x+\frac{1}{2}\right)$ for x > 0. [3]

c) Prove that $\beta(m,n)=\dfrac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$ for m, n > 0. [4]

5. a) Show that $\int_1^{\infty}\frac{\sin x}{x^p}dx$ converges for p > 0. [3]

b) Examine the convergence of the improper integral $\int_0^{\pi}\frac{dx}{\cos\alpha-\cos x}$, $0\le\alpha\le\pi$. [3]

c) If f and g be two positive valued functions in [a,b] such that both have infinite discontinuity at 'a' only, both are integrable on $[a+\varepsilon,b]$, $0<\varepsilon<b-a$ and $\lim_{x\to a+}\frac{f(x)}{g(x)}=\ell$, where ℓ is a non zero finite number, prove that $\int_a^b f(x)dx$ and $\int_a^b g(x)dx$ converge or diverge together. [4]

Unit – II

[*Answer **any two** questions*]

6. a) If a, b are integers; not both zero, then there exists integers u, v such that $\gcd(a,b)=au+bv$. [4]

b) Prove that the total number of positive divisors of a positive integer n is odd if and only if n is a perfect square. [4]

c) Find the number of zeros at the right end of the integer (141)!. [2]

7. a) Let a > 1, be a positive integer. Prove that $n\mid\phi(a^n-1)$, where $n\in\mathbb{N}$. [4]

b) If p be a prime then prove that $(p-1)!+1\equiv 0 \pmod p$ [3]

c) For any prime p > 3, prove that 13 divides $10^{2p}-10^p+1$. [3]

8. a) Solve the system of lineal congruences $x\equiv 2(\text{mod}\,5)$, $x\equiv 3(\text{mod}\,7)$, $x\equiv 5(\text{mod}\,8)$. [4]

b) State and prove Möbius inversion formula. [4]

c) Let $p=p_1p_2...p_n$, the product of first n primes. Show that p+1 and p–1 are not perfect squares. [2]

Group – B

Unit – I

[*Answer **any three** questions*]

9. a) For n events A_1, A_2, ... A_n connected with random experiment E, show that $P(A_1A_2...A_n)\ge\sum_{i=1}^{n}P(A_i)-(n-1)$. Hence show that $P\left(\sum_{i=1}^{n}A_i\right)\le\sum_{i=1}^{n}P(A_i)$. [3+2]

b) A and B throw a pair of fair dice. A wins, if he throws 6 before B throw 7 and B wins, if he throws 7 before A throws 6. If A begins, what is the chance of B's winning? Also find the chance of B's winning when B begins. [5]

10. a) If X is a Poisson variate with parameter μ, show that $P(X\le n)=\frac{1}{\lfloor n}\int_{\mu}^{\infty}e^{-x}x^n dx$, when $n\in\mathbb{N}$. [4]

b) The probability density function of a continuous random variable X is given by

$$f_x(x)=\begin{cases}\dfrac{1}{\sqrt{2\pi}}\cdot\dfrac{e^{-\frac{1}{2}}(\ell n x)^2}{x}, & 0<x<\infty\\ 0, & \text{elsewhere}\end{cases}$$

Show that the probability distribution of the random variable ℓnX is standard normal. [3]

c) Prove that the distribution function $F_{x,y}$ of a random variable (X,Y) is non-decreasing in both the variables x and y. [3]

11. a) Find the probability P_N that a natural number choosen at random from the set {1, 2, 3, ...N} is divisible by a fixed natural number K. [4]

b) Two people agree to meet at a definite place between 12 and 1 o'clock with the understanding that each will wait 20 minutes for the other. What is the probability that they will meet. [4]

c) Show that second order moment about any point is minimum when taken about the mean. [2]

12. a) If X is a binomial (n, p) variate, then prove that $\mu_{k+1} = p(1-p)\left(nk\mu_{k-1} + \frac{d\mu_k}{dp}\right)$, where μ_k is the kth central moment. [5]

b) Find the characteristic function of a normal (m, σ) distribution. Hence show that if $X_1, X_2, ..., X_n$ are n independent normal variates with parameters $(m_1, \sigma_1), (m_2, \sigma_2), ...(m_n, \sigma_n)$ respectively, then the random variable $S_n = X_1 + X_2 + ... + X_n$ is also a normal variate with parameters (m, σ), where $m = m_1 + m_2 + ... + m_n$ and $\sigma = \sqrt{\sigma_1^2 + \sigma_2^2 + ... + \sigma_n^2}$. [2+3]

13. a) If θ be the acute angle between the least square regression lines then show that $\tan\theta = \frac{1-\rho^2}{|\rho|} \cdot \frac{\sigma_x \sigma_y}{\sigma_x^2 + \sigma_y^2}$. Discuss the cases when $\rho = 0$ and $\rho = \pm 1$. (the symbols are of usual conversion). [3+1+1]

b) A random variable X has probability density function

$$f(x) = 12x^2(1-x); \; 0 < x < 1$$
$$= 0 \quad ; \text{elsewhere}$$

compute $P(|X - m| \geq 2\sigma)$, $\left.\begin{matrix} m = E(x) \\ \sigma = +\sqrt{\text{var}(X)} \end{matrix}\right\}$

Compare it with the limit given by Tchebycheff's inequality. [3+2]

<u>Unit – II</u>

[*Answer **<u>any two</u>** questions*]

14. a) Test for the convergence of $\{z_n\}_n$ and $\{\text{Arg}\, z_n\}_n$ where $z_n = -2 + i\frac{(-1)^n}{n^2}$. [3]

b) Find the radius of convergence of the power series $\sum_{n=0}^{\infty} (\cos in) z^n$. [3]

c) Let f(z) and g(z) be two analytic functions on a region D such that $\text{Re}(f) = \text{Re}(g) \; \forall z \in D$. Show that there is a constant K such that $f(z) - g(z) = K, \; \forall z \in D$. [4]

15. a) If $f(x+iy) = Re^{i\phi}$ be analytic, show that $\frac{\partial R}{\partial x} = R\frac{\partial \phi}{\partial y}$. [3]

b) If $f(z) = u(x, y) + iv(x, y)$ be analytic, show that $f(z) = 2u\left(\frac{z}{2}, -\frac{iz}{2}\right)$ + constant. [3]

c) Let the function $f(z) = u(x, y) + iv(x, y)$ be defined in some neighbourhood of $z_0 = x_0 + iy_0 (x_0, y_0 \in \mathbb{R})$. Suppose the first order partial derivatives of u and v are continuous at (x_0, y_0) and satisfy the Cauchy-Riemann equations at that point. Prove that f' exists at z_0. [4]

16. a) Answer either (i) or (ii) [4]

i) Assuming that the radius of convergence of the power series $\sum_n \frac{z^{4n}}{4n+1}$ as 1, show that the series converges at all points on the circle of convergence except four points which you have to mention.

ii) Let $f(z)=\sum_{n=0}^{\infty} a_n z^n$ and $g(z)=\sum_{n=0}^{\infty} b_n z^n$ converge for $|z|<\rho_1$ and $|z|<\rho_2$ respectively. Let $0<\delta<\min\{\rho_1,\rho_2\}$. If $f(z_K)=g(z_K)$ for a sequence of distinct points, $\{z_K\}_K$ in $0<|z|<\delta$ such that $z_K \to 0$ as $K\to\infty$, then show that $a_n = b_n$ for every n.

b) Let f(z) be analytic in a region D. If arg f is constant in D, show that f(z) is constant in D. [3]

c) If $f(z)=u(x,y)+iv(x,y)$ be analytic function. Find harmonic conjugate of u, where $u=e^x(x\cos y - y\sin y)$. [3]

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